The Standard Conjectures

Up to now the construction of motives has not relied on any unproven assumptions. But we need to remind ourselves of the goal, namely to get a universal cohomology theory. To do this, one would need to address several conjectures of Grothendieck:

To explain the conjectures, fix a Weil cohomology theory H with coefficients from a characteristic zero field F. Recall that such a cohomology theory comes equipped with a cycle map $cl_X : CH^i(X)_{\mathbb{Q}} \longrightarrow H^{2i}(X)$, and the surjective image of cl_X in $H^{2i}(X)$ are called algebraic classes.

1. The Künneth conjecture $C(X)$

Let $\Delta(X) \subset X \times X$ be the diagonal, and consider $cl_{X\times X}(\Delta(X)) \in H^{2d}(X\times X)$. By the assumption on H, we have a Künneth decomposition:

$$H^{2d}(X\times X) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^{i}(X).$$

Denote by Δ_i the i^{th} component of $\Delta(X)$ in this direct sum. Then:

Conjecture $C(X)$: The Künneth components Δ_i are algebraic.

This conjecture is known in a handful of cases. First when the variety admits some "algebraic cell decomposition", also known for curves, surfaces, and abelian varieties. Katz + Messing have also proven it for $k = \mathbb{F}_q$. For most of these claims, see Kleiman's article, "Algebraic Cycles and the Weil Conjectures".

Another tidbit, over $k = \mathbb{C}$, $C(X)$ would follow from the Hodge conjecture.

2. Conjectures of Lefschetz Type

Choose some explicit projective embedding of $X_d \hookrightarrow \mathbb{P}^n$, and Y a hyperplane section. Then we have the Lefschetz operator:

$$L : H^i(X) \longrightarrow H^{i+2}(X), \quad \alpha \longmapsto \alpha \cup cl_X(Y).$$

Since we have assumed hard Lefschetz, this gives isomorphisms $L^i : H^{d-i} \xrightarrow{\sim} H^{d+i}$. We then define

$$\Lambda = (L^{i+2})^{-1} \circ L \circ L^i : H^i(X) \to H^{i-2}(X),$$

i.e. by the following diagram

$$\begin{tikzcd} H^{d-i}(X) \arrow[r, "L^i"] \arrow[d, dashed, "\Lambda"'] & H^{d+i}(X) \arrow[d, "L"] \\ H^{d-i-2}(X) \arrow[r, "L^{i+2}"] & H^{d+i+2} \end{tikzcd} \qquad 0 \le i \le r$$

and similar for other bounds.

Alternatively we can define Λ using primitive elements $P^i(X) = \mathrm{Ker}(L^{d-i+1} : H^i \to H^{2d-i+2})$ we can decompose $H^i(X) = P^i(X) \oplus LH^{i-2}(X)$, hence every element $a \in H^i(X)$ has a unique primitive decomposition:

$$a = \sum_{j \geq \max(i-r,0)} L^j a_j \quad , \quad a_j \in P^{i-2j}(X).$$

Then define $\Lambda a = \sum_{j \geq \max(i-r,1)} L^{j-1} a_j$. Note that Λ is almost an inverse to L.

Further, the linear map $\Lambda : H^\bullet(X) \to H^\bullet(X)$ comes from a <u>topological</u> correspondence, i.e. an element of $H^\bullet(X \times X)$, and so

<u>Conjecture B(X):</u> The correspondence Λ is algebraic.

As before, if $k = \mathbb{C}$, then $B(X)$ would follow from the Hodge conjecture. In general it is known to hold if X is a curve, surface with $h^1(X) = 2 \cdot \dim \mathrm{Pic}^0(X)$, an abelian variety, or a generalized flag manifold. Further $B(X)$ is independent of $X \hookrightarrow \mathbb{P}^n$ and the hyperplane section.

Consider the commutative diagram:

$$\begin{tikzcd}
H^{2i}(X) \arrow[r, "L^{d-2i}", "\sim"'] & H^{2d-2i}(X) \\
A^i(X) \arrow[u, "cl_X"] \arrow[r, "\text{injective by Hard Lefschetz}"'] \arrow[d, equal] & A^{d-i}(X) \arrow[u, "cl_X"'] \arrow[d, equal] \\
\mathrm{Im}(cl_X) & \mathrm{Im}(cl_X)
\end{tikzcd}$$

<u>Conjecture A(X, L):</u> $A^i(X) \to A^{d-i}(X)$ is an isomorphism. Alternatively, the cup product $A^i(X) \times A^{d-i}(X) \to \mathbb{Q}$ is an isomorphism.

Note that $B(X) \Rightarrow A(X, L)$, and if you believe Grothendieck $A(X,L) \Rightarrow B(X)$.

3. Conjectures of Hodge Type

Recall that we have defined primitive cohomology as: $P^i(X) = \mathrm{Ker}(L^{d-i+1} : H^i(X) \to H^{2d-i+2}(X))$ and hence we can talk about the primitive algebraic classes:

$$A^i_{prim}(X) = A^i(X) \cap P^{2i}(X).$$

Then the cup product induces a new pairing $A^i_{prim}(X) \times A^i_{prim}(X) \to \mathbb{Q}$, by sending $(x,y) \mapsto (-1)^i \, \mathrm{Tr} \circ (L^{d-2i}(x) \cup y) \in \mathbb{Q}$.

<u>Conjecture Hdg(X):</u> The pairing above is positive definite.

If $k = \mathbb{C}$, Hodge theory gives a proof, as a comparison with the Betti-cohomology and the Riemann Bilinear Relations proves it. Its also known for surfaces over any field.

Some relations: • $D(X) \Rightarrow A(X,L)$, as then $A^i(X) = Z(X)/Z_{num}(X)$, and the pairing is nondegenerate by definition.

- If $Hdg(X)$, then $D(X) \Longleftrightarrow A(X,L)$.
- $B(X) + Hdg(X) \Rightarrow D(X)$.
- $B(X) + Hdg(X) \Rightarrow Mot_{num}$ is abelian semisimple. This is actually already true.

In particular all of them would imply the existence of a universal Weil cohomology theory, which would be given by Mot_{num}. The surprising fact is known as Jannsen's theorem

Thm (Jannsen): The following are equivalent:
1) $Mot_\sim$ is abelian semisimple,
2) $\sim = \sim_{num}$,
3) for all $X_d \in SmProj_k$, the F-algebra $Corr^0_\sim(X,X)_F$ is a finite dimensional semi-simple F-algebra.

See Murre for the proof.